

Thermodynamics of the Variable Modified Chaplygin gas

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Abstract

A cosmological model with a new variant of Chaplygin gas obeying an equation of state(EoS), $P = A\rho - \frac{B}{\rho^\alpha}$ where $B = B_0 a^n$ is investigated in the context of its thermodynamical behaviour. Here B_0 and n are constants and a is the scale factor. We show that the equation of state of this ‘Variable Modified Chaplygin gas’ (VMCG) can describe the current accelerated expansion of the universe. Following standard thermodynamical criteria we mainly discuss the classical thermodynamical stability of the model and find that the new parameter, n introduced in VMCG plays a crucial role in determining the stability considerations and should always be *negative*. We further observe that although the earlier model of Lu explains many of the current observational findings of different probes it fails the desirable tests of thermodynamical stability. We also note that for $n < 0$ our model points to a phantom type of expansion which, however, is found to be compatible with current SNe Ia observations and CMB anisotropy measurements. Further the third law of thermodynamics is obeyed in our case. Our model is very general in the sense that many of earlier works in this field may be obtained as a special case of our solution. An interesting point to note is that the model also apparently suggests a smooth transition from the big bang to the big rip in its whole evaluation process.

KEYWORDS : cosmology;Chaplygin gas;thermodynamics

PACS : 98.80.-k,98.80.Es,95.30.Tg,05.70.Ce

1. Introduction

Following the high redshift supernovae data in the last decade [1, 2] we know that when interpreted within the framework of the standard FRW type of universe (homogeneous and isotropic) we are left with the only alternative that the universe is now going through an accelerated expansion with baryonic matter contributing only 5% of the total budget. Later data from CMBR studies [2] further corroborate this conclusion which has led a vast chunk of cosmology community ([3] and references therein) to embark on a quest to explain the cause of the acceleration. In fact the studies on accelerated expansion and its possible interpretations from different angles have been reigning the research paradigm for the last few decades. The teething problem now confronting researchers in this field is the identification of the mechanism that triggered the late inflation. But as they are already discussed

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extensively in the literature (we are sparing the readers here to repeat once again those arguments) all the alternatives face serious theoretical problems in some form or other.

In the literature a good number of dark energy models are proposed but little is precisely known about its origin. Nowadays, the dark energy problem remains one of the major unsolved problems of theoretical physics [4]. On the way of searching for possible solutions of this problem various models are explored during last few decades referring to e.g. new exotic forms of matter (*e.g.* quintessence) [5], phantom [6], holographic models [7], string theory landscape [8], Born-Infeld quantum condensate [9], modified gravity approaches [10], inhomogeneous spacetime [11], higher dimensional space time [12] etc.

While the above mentioned alternatives to address the observed acceleration of the current phase have both positive and negative aspects a number of papers have come up taking into account the Chaplygin gas [13–15] as a new form of matter field to simulate unified dark energy and dark matter model. It is presumed that for gravitational attraction, the dark matter component is responsible for galaxy structure formation while dark energy provides necessary repulsive force for current accelerated expansion.

Motivated by the desire to explain away the observational fallouts better and better the form of the equation of state (in short, EoS) of matter is later generalised in stages first through the addition of an arbitrary constant with an exponent over the mass density, generally referred to as generalised Chaplygin gas (GCG) [3]. Barring the serious disqualification *e.g.*, it violates the time honoured principle of energy conditions, it is fairly successful to interpret the observational results coming out of gravitational lensing or recent CMBR and SNe data in varied cosmic probes via the fine tuning of the value of the newly introduced arbitrary constant. The form of EoS is again modified through the addition of an ordinary matter field, which is termed in the literature as modified Chaplygin gas (MCG) [16–18], claiming an even better match with observational results.

It may be appropriate at this stage to call attention to a recent work by Guo and Zhang [19,20] where a very generalised form of the Chaplygin gas relation is invoked, assuming the constant B to depend on the scale factor a . Taking, *e.g.*, $B = B_0 a^{-n}$, it is shown that for a very large value of the scale factor the model interpolates between a dust-dominated phase and a quiescence phase (i.e., dark energy with a constant equation of state) [12,14] given by $\mathcal{W} = -1 + \frac{n}{6}$.

As mentioned earlier, although different variants of dark energy models as well as Chaplygin type of gas models are brought in the cosmological arena as also its associated success to explain the observations coming out of different cosmic probes it has not escaped our notice that scant attention has been paid so far to address the important issue if all the so called perfect gas models are atleast thermodynamically stable. Otherwise they would lose the claim to be treated as a physically realistic system. Following this there has been of late a resurgence of interests among workers to address their queries to this aspect of the problem. Recently Santos et al have studied the thermodynamical stability in generalised [21] and modified Chaplygin [22] gas model on the basis of standard prescription [23] where both (i) $(\frac{\partial P}{\partial V})_S < 0$, $(\frac{\partial P}{\partial V})_T < 0$ and (ii) $c_V > 0$ are satisfied simultaneously.

In the present work we attempt to generalise their results to Gou-Zhang formal-

ism as also to clarify the issues concerning thermodynamics of the Chaplygin gas. We also investigate the nature of physical parameters such as pressure (P), effective equation of state (\mathcal{W}), deceleration parameter (q) etc. on the basis of thermodynamics. Later many thermal quantities are derived as functions of temperature or volume. In this case, we also show as consistency check that the third law of thermodynamics is satisfied by the new form of the Chaplygin gas. For the generalised Chaplygin gas we expect to have almost similar behaviour as the Chaplygin gas equations did show. Further, we see that the Chaplygin gas which shows a unified picture of dark matter and energy cools down through the adiabatic expansion of the universe without any critical point. Returning to the stability criterion of the Chaplygin gas we find a striking difference from the analysis of Santos et al. In our case of variable modified Chaplygin gas (VMCG) [24] we find that the stability depends critically on the new parameter n , as introduced by Guo et al. We also notice that stability decreases with the increase of magnitude of n , which apparently disfavors the formalism of Guo-Zhang [19]. In the section on acoustic velocity we interestingly note that unlike the previous Chaplygin gas models where the squared sound velocity is always positive definite, the velocity here depends on the value of the newly introduced parameter n . But as noted earlier the stability criteria dictates that n should be negative which, however, makes the squared velocity also negative at the late stage of evolution. This means that the perfect fluid model for Guo's variant of the chaplygin gas is classically unstable and is in line with the results obtained earlier by Myung [25] while dealing with the holographic interpretation for Chaplygin gas and tachyon. The paper ends with a short discussion.

2. Formalism

The line element corresponding to spatially flat FRW spacetime is given by

$$ds^2 = dt^2 - a^2(t) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (1)$$

where $a(t)$ is the scale factor. In this work we consider the following equation of state

$$P = A\rho - \frac{B}{\rho^\alpha}. \quad (2)$$

A and B are positive constants. As discussed in the introduction we have taken $B = B_0 V^{-\frac{n}{3}}$ where n is an arbitrary constant and B_0 an absolute constant. Here P corresponds to the pressure and ρ the energy density of that fluid such that

$$\rho = \frac{U}{V}, \quad (3)$$

where U and V are the internal energy and volume filled by the fluid respectively.

We try to find out the energy density U and pressure P of Variable Modified Chaplygin gas as a function of its entropy S and volume V . From general thermodynamics, one has the following relationship

$$\left(\frac{\partial U}{\partial V} \right)_S = -P. \quad (4)$$

With the help of the above equations we get

$$\left(\frac{\partial U}{\partial V}\right)_S = B_0 V^{-\frac{n}{3}} \frac{V^\alpha}{U^\alpha} - A \frac{U}{V}, \quad (5)$$

which, on integration yields

$$U = \left[\frac{3B_0(1+\alpha)V^{\frac{3(1+\alpha)-n}{3}}}{3(A+1)(1+\alpha)-n} + \frac{c}{V^{A(1+\alpha)}} \right]^{\frac{1}{1+\alpha}}. \quad (6)$$

The parameter c is the integration constant, which may be a universal constant or a function of entropy S only. Now we rewrite the above equation in the following form

$$U = \left[\frac{B_0(1+\alpha)V^{-\frac{n}{3}}}{N} \right]^{\frac{1}{1+\alpha}} V \left[1 + \left(\frac{\epsilon}{V} \right)^N \right]^{\frac{1}{1+\alpha}}, \quad (7)$$

where $N = \frac{3(A+1)(1+\alpha)-n}{3} > 1$ for $(A+1)(1+\alpha) > \frac{n}{3}$ for real U and

$$\epsilon = \left[\frac{3(A+1)(1+\alpha)-n}{3B_0(1+\alpha)} c \right]^{\frac{1}{N}} = \left[\frac{Nc}{B_0(1+\alpha)} \right]^{\frac{1}{N}}, \quad (8)$$

which has a dimension of volume. Now the energy density ρ of the VMCG comes out to be

$$\rho = \left[\frac{B_0(1+\alpha)V^{-\frac{n}{3}}}{N} + V^{-N-\frac{n}{3}} c \right]^{\frac{1}{1+\alpha}} \quad (9a)$$

$$= \left[\frac{B_0(1+\alpha)V^{-\frac{n}{3}}}{N} \right]^{\frac{1}{1+\alpha}} \left[1 + \left(\frac{\epsilon}{V} \right)^N \right]^{\frac{1}{1+\alpha}}. \quad (9b)$$

From what has been discussed above we like to obtain the expression of relevant physical quantities and investigate their behaviour.

(a) Pressure :

Using equations (2) and (9b) the pressure P of the VMCG may also be determined as a function of entropy S and volume V in the following form

$$P = - \left[\frac{B_0(1+\alpha)V^{-\frac{n}{3}}}{N} \right]^{\frac{1}{1+\alpha}} \left(\frac{N}{1+\alpha} \right) \frac{\left[1 - \frac{A(1+\alpha)}{N} \left\{ 1 + \left(\frac{\epsilon}{V} \right)^N \right\} \right]}{\left[1 + \left(\frac{\epsilon}{V} \right)^N \right]^{\frac{\alpha}{1+\alpha}}}. \quad (10)$$

The equation (10) gives a very general expression of pressure. Under special conditions

i) For $n = 0$ & $A = 0$, the equation (10) reduces to

$$P = - \frac{(B_0)^{\frac{1}{1+\alpha}}}{\left\{ 1 + \left(\frac{c}{B_0 V} \right)^{1+\alpha} \right\}^{\frac{\alpha}{1+\alpha}}}, \quad (11)$$

which is, as expected, similar to generalised Chaplygin gas (GCG) model [21].

ii) For $n = 0$ & $A \neq 0$, we get the modified Chaplygin gas (MCG) model with pressure given by [22]

$$P = - \left(\frac{B_0}{1+A} \right)^{\frac{1}{1+\alpha}} \frac{\left\{ 1 + A - A \left[1 + \left\{ \frac{(1+A)c}{B_0 V} \right\}^{(1+A)(1+\alpha)} \right] \right\}}{\left[1 + \left\{ \frac{(1+A)c}{B_0 V} \right\}^{(1+A)(1+\alpha)} \right]^{\frac{\alpha}{1+\alpha}}}. \quad (12)$$

iii) For $A = 0$, but $n \neq 0$, the model represents the Variable Chaplygin gas (VCGC) [26] and equation (10) becomes

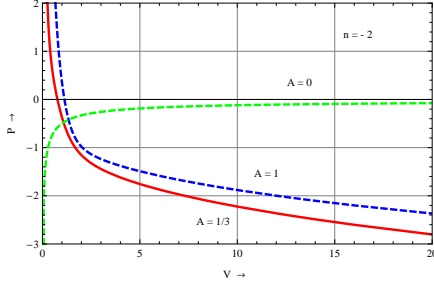
$$P = - \left(B_0 V^{-\frac{n}{3}} \right)^{\frac{1}{1+\alpha}} \left[\frac{N}{(1+\alpha) \left\{ 1 + \left(\frac{\epsilon}{V} \right)^N \right\}} \right]^{\frac{\alpha}{1+\alpha}}. \quad (13)$$

Again if we put $\alpha = 1$ in equation (13), we get $P = - \left[\frac{N B_0 V^{-\frac{n}{3}}}{2 \left\{ 1 + \left(\frac{\epsilon}{V} \right)^N \right\}} \right]^{\frac{1}{2}}$ which is identical with our previous work [27] when we consider the Variable Chaplygin gas (VCG) model.

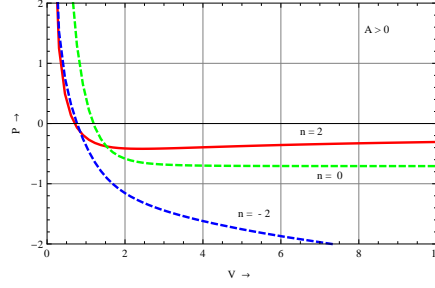
The pressure may have positive or negative values, depending on the magnitude of both A & n and also on volume V (fig - 1). For $P = 0$, let V is denoted by V_c , which is given by

$$V_c = \epsilon \left[\frac{3A(1+\alpha)}{3(1+\alpha) - n} \right]^{\frac{1}{N}}, \quad (14)$$

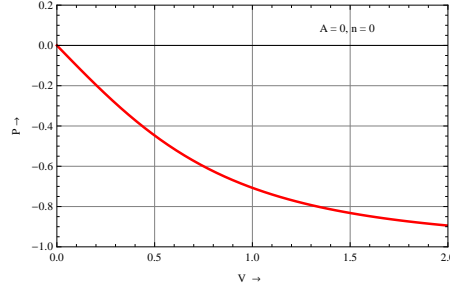
which restricts n as $n < 3(1+\alpha)$. Initially, *i.e.*, $V < V_c$, P is positive, which indicates a radiation dominated universe. For $V = V_c$, $P = 0$ and $V > V_c$, P is negative pointing to a state of accelerating universe. This is an interesting result showing that V_c introduces a new scale in the analysis, beyond which a dust dominated universe enters the acceleration era. With expansion an initially decelerating universe tends to reverse its motion and prepares to accelerate when its volume crosses a critical value designated by V_c . It is also to be noted that for physically realistic values of constants both V_c and ϵ are of the same order of magnitude *i.e.* ϵ also signifies a volume scale beyond which accelerating era commences. So $V \gg \epsilon$ represents a very large volume and $V \ll \epsilon$ the reverse. In what follows we will see that this statement has significant cosmological implications.



(a) The graphs clearly show that pressure P can be both positive and negative depending on the value of A . For $A = 0$, P is always negative, i.e. Chaplygin type gas with n which is also in accord with the equation (2).



(b) It show that pressure P is positive for small value of V and negative for large V when $A > 0$. As the value of n tends towards negative, P becomes more negative.



(c) For $A = 0$, $n = 0$, P is always negative, chaplygin type case.

Figure 1: The variation of P and V for different values of A and n . Here we have taken $B_0 = 1$, $\alpha = 1$ & $c = 1$ for constant S .

(b) Caloric EoS:

Now using the expression (9b) and (10) we get the caloric equation of state parameter

$$\mathcal{W} = \frac{P}{\rho} = A - \left(\frac{N}{1 + \alpha} \right) \frac{1}{1 + \left(\frac{\epsilon}{V} \right)^N}. \quad (15)$$

As the last EoS is very involved in nature it is very difficult to extract much physics out of it. So we look forward to its extremal cases as:

1. For small volume, $V \ll \epsilon$, we get from equation (15) that

$$P \approx A\rho. \quad (16)$$

This is a barotropic equation of state. In this case no influence of n on small volume.

2. For large volume, $V \gg \epsilon$, the equation (15) reduces to

$$\mathcal{W} \approx -1 + \frac{n}{3(1 + \alpha)}. \quad (17)$$

Equation (14) shows that $n < 3(1 + \alpha)$. So there are three possibilities for \mathcal{W} depending on the signature of n as (i) $n > 0$, $\mathcal{W} > -1$, here the caloric equation

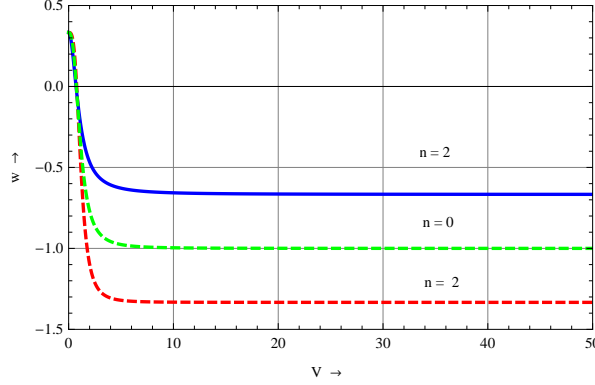


Figure 2: The variation of \mathcal{W} and V for different values of n . The values of constants are taken as $A = \frac{1}{3}$, $B_0 = 1$, $\alpha = 1$, $c = 1$.

of state results in a quiescence type and big rip is avoided in this case, (ii) $n = 0$, $\mathcal{W} = -1$, *i.e.*, we get Λ CDM. (iii) $n < 0$, $\mathcal{W} < -1$ represents phantom like universe. This is compatible with recent observational results [28]. Influence of n is prominent in this case.

Initially, *i.e.*, when volume is sufficiently small, $\mathcal{W} > 0$. As V increases to V_c , \mathcal{W} tends to zero. When $V = V_c = \epsilon \left[\frac{3A(1+\alpha)}{3(1+\alpha)-n} \right]^{\frac{1}{N}}$, $\mathcal{W} = 0$. Again V increases and \mathcal{W} becomes negative. We have seen from fig-2 that $n = 0$, $\mathcal{W} = -1$, *i.e.*, Λ CDM model which is currently fashionable. But for $n > 0$ we get $0 > \mathcal{W} > -1$ for large V . Similar result was shown earlier in higher dimensional case also [12]. We do not discuss much about $n > 0$ because later we will show from thermodynamical stability considerations that the value of $n \leq 0$.

(c) Deceleration parameter:

Now using equation (15) we calculate the deceleration parameter of the VMCG fluid as

$$q = \frac{1}{2} + \frac{3P}{2\rho} = \frac{1}{2} + \frac{3}{2} \left\{ A - \left(\frac{N}{1+\alpha} \right) \frac{1}{1 + \left(\frac{\epsilon}{V} \right)^N} \right\}, \quad (18)$$

For mathematical simplicity here also we discuss the extreme cases.

1. For small volume, $V \ll \epsilon$, it gives

$$q \approx \frac{1}{2} + \frac{3}{2}A, \quad (19)$$

i.e., q is positive, universe decelerates for small V .

2. For large volume, $V \gg \epsilon$, the equation (18) reduces to

$$q \approx -1 + \frac{n}{2(1+\alpha)}. \quad (20)$$

Initially, *i.e.*, when volume is very small there is no effect of n on q , here q is positive, universe decelerates. But for large volume, q is negative and this depends on the value of n also. For *flip* in velocity to occur the flip volume (V_f) becomes

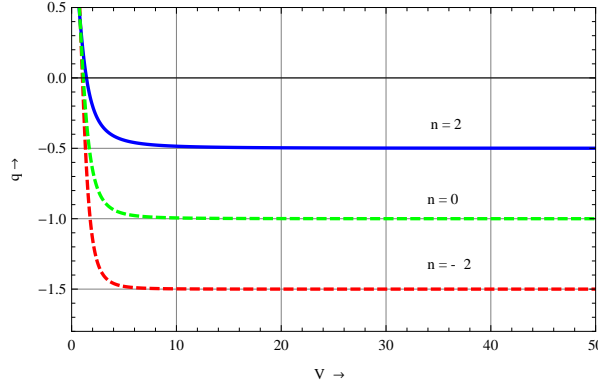


Figure 3: The variation of q and V for different values of n . We have considered here $A = \frac{1}{3}$, $B_0 = 1$, $\alpha = 1$ & $c = 1$.

$$V_f = \epsilon \left[\frac{(1+3A)(1+\alpha)}{2(1+\alpha)-n} \right]^{\frac{1}{N}}. \quad (21)$$

Therefore $n < 2(1+\alpha)$, which interestingly does not violate our previous restriction on n . A little analysis of the above equation shows that for V_f to have real value $n < 2(1+\alpha)$, otherwise there will be no *flip*. This also follows from the fig - 3 where only $n < 4$ allows *flip* (for $\alpha = 1$). Alternatively the inequality $n < 2(1+\alpha)$ may be treated as our acceleration condition. Thus $V < V_f$, we get deceleration; and $V > V_f$, acceleration occur.

It has not also escaped our notice that we are here getting two scales (V_c and V_f) for volume where apparently acceleration flip occurs. Again we know that in a FRW cosmology for flip to occur pressure should not only be negative but its magnitude should be less than $\frac{\rho}{3}$ (i.e., $\rho + 3P < 0$) as well. So in our case one should get $V_c < V_f$, which also follows from their expressions (14) and (21). In this connection it may also be mentioned that with $V = V_f$, $\rho + 3P = 0$ as expected.

(d) Velocity of Sound:

Let us consider v_s be the velocity of sound, then using equation (10) we can write

$$v_s^2 = \left(\frac{\partial P}{\partial \rho} \right)_s = A + \frac{N\alpha}{(1+\alpha) \left\{ 1 + \left(\frac{\epsilon}{V} \right)^N \right\}} - \frac{nN}{n + (3N+n) \left(\frac{\epsilon}{V} \right)^N}. \quad (22)$$

Since sound speed should be $0 < \left(\frac{\partial P}{\partial \rho} \right)_s < 1$, now our analysis shows ,

1. For small volume, i.e., at early universe, the equation (22) leads to $0 < A < 1$, and this limit includes $A = \frac{1}{3}$, the radiation dominated universe.

2. For large volume, the equation (22) reduces to

$$v_s^2 = -1 + \frac{n}{3(1+\alpha)}. \quad (23)$$

The equation (23) does not depend on A , it depends on n and α only. In what follows we shall see that from the thermodynamical stability condition the value of $n < 0$, leading to a phantom universe [28]. Moreover the equation (23) gives an imaginary speed of sound for $\alpha > 0$, leading to a perturbative cosmology. One need not be too sceptic about it because it favours structure formation [29].

It may not be out of place to draw some correspondence to a recent work by Y.S. Myung [25] where a comparison is made between holographic dark energy, Chaplygin gas, and tachyon model with constant potential. We know that their squared speeds are crucially important to determine the stability of perturbations. They found that the squared speed for holographic dark energy is always negative when imposing the future event horizon as the IR cutoff, while those for original Chaplygin gas and tachyon are always non-negative. This is in sharp contrast to our variant (VMCG) of the Chaplygin gas model where we observe that depending on the signature of n the squared velocity may be both positive or negative. However as discussed earlier a non negative value of n is clearly ruled out from thermodynamic stability considerations. This points to the fact that the perfect fluid model for VMCG dark energy model is classically unstable like the holographic interpretation for Chaplygin gas and tachyon and hence problematic in the long run.

(e) Thermodynamical Stability:

To verify the thermodynamic stability conditions of a fluid along its evolution, it is necessary (a) to determine if the pressure reduces both for an adiabatic and isothermal expansion [23] $(\frac{\partial P}{\partial V})_S < 0$ & $(\frac{\partial P}{\partial V})_T < 0$ and (b) also to examine if the thermal capacity at constant volume, $c_V > 0$.

Using equations (2) and (10) we get

$$\left(\frac{\partial P}{\partial V}\right)_S = \frac{P}{3V(1+\alpha)} \frac{A(1+\alpha) \left\{ n \left[1 + \left(\frac{\epsilon}{V}\right)^N \right] + 3N \left(\frac{\epsilon}{V}\right)^N \right\} + N \left\{ \frac{3N\alpha \left(\frac{\epsilon}{V}\right)^N}{\left[1 + \left(\frac{\epsilon}{V}\right)^N \right]} - n \right\}}{N - A(1+\alpha) \left[1 + \left(\frac{\epsilon}{V}\right)^N \right]}. \quad (24)$$

Now we have to examine the negativity of $(\frac{\partial P}{\partial V})_S$. As this expression (24) is so involved we can not make any conclusion considering this equation as a whole. Our analyses naturally split into two parts :

1. Firstly, we have considered small volume, $V \ll \epsilon$, where equation (24) gives $(\frac{\partial P}{\partial V})_S \approx -(1+A)\frac{P}{V}$. We get from previous analysis that at the early stage of evolution $P \approx A\rho$, therefore, in this case $(\frac{\partial P}{\partial V})_S \approx -A(1+A)\frac{\rho}{V}$ which is independent of n but very much depends on A and at this stage of evolution $(\frac{\partial P}{\partial V})_S < 0$.
2. For large volume, $V \gg \epsilon$, the equation (24) reduces to $(\frac{\partial P}{\partial V})_S \approx -\frac{nP}{3V(1+\alpha)}$. Since P is negative at the late stage of evolution, so n must be negative to make $(\frac{\partial P}{\partial V})_S < 0$. We have seen that the dependence of n is prominent at the later case. From fig - 4 we get the similar type of conclusion.

Now we discuss some special cases to constrain the parameters used here.

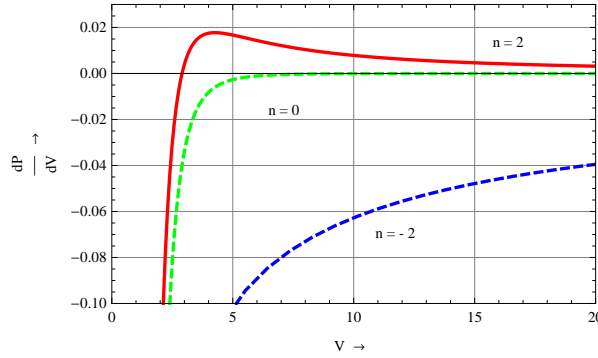


Figure 4: The variation of $(\frac{\partial P}{\partial V})_S$ and V for different values of n . The nature of graphs shows that for $n \leq 0$, $(\frac{\partial P}{\partial V})_S < 0$ throughout the evolution unlike $n > 0$ where $(\frac{\partial P}{\partial V})_S < 0$ only at the early stage. We have taken $A = 1, B_0 = 1, \alpha = 1, c = 1$.

(i) The simultaneous conditions $n = 0, \alpha = 0$ and $A = 0$ must be discarded because it will place a severe restriction on the stability of this fluid, in such a case $(\frac{\partial P}{\partial V})_S = 0$ and the pressure will remain the same through any adiabatic change of volume. However B_0 here behaves like a Cosmological Constant. We get the de-Sitter type of metric.

(ii) Again for $\alpha = 0, A = 0$ and $n \neq 0$, $(\frac{\partial P}{\partial V})_S = \frac{n}{3}B_0V^{-(1+\frac{n}{3})}$, i.e., for $n < 0$, $(\frac{\partial P}{\partial V})_S < 0$ which is in disagreement with the previous work of Santos *et al* [21] where simultaneously $\alpha = 0, A = 0$ is not possible. Relevant to mention that, $n \leq 0$ implies that the pressure goes more and more negative with volume. This agrees with the observational results [19].

(iii) When $A = 0, n = 0$ and $\alpha \neq 0$, the equation (24) reduces to Santos's work [21]. In this case $(\frac{\partial P}{\partial V})_S$ will be

$$\left(\frac{\partial P}{\partial V}\right)_S = \alpha \frac{P}{V} \left(\frac{\epsilon}{V}\right)^{1+\alpha} \left\{1 + \left(\frac{\epsilon}{V}\right)^{1+\alpha}\right\}^{-1}, \quad (25)$$

which gives $(\frac{\partial P}{\partial V})_S < 0$ for $\alpha > 0$. It also agrees with the work of Santos *et al* [21] in this field.

(iv) when $\alpha = 0, n = 0$ and $A \neq 0$, $(\frac{\partial P}{\partial V})_S = -\frac{AB_0}{V} \left(\frac{\epsilon}{V}\right)$, i.e., $(\frac{\partial P}{\partial V})_S < 0$ which implies simultaneously $A > 0$ and $B_0 > 0$.

(v) For $A \neq 0$ but $\alpha \neq 0$ & $n = 0$ the equation (24) reduces to

$$\left(\frac{\partial P}{\partial V}\right)_S = \frac{P}{V} \frac{\left(\frac{\epsilon}{V}\right)^N}{1 - \frac{A}{1+A} \left\{1 + \left(\frac{\epsilon}{V}\right)^N\right\}} \left[A + \frac{\alpha(1+\alpha)}{1 + \left(\frac{\epsilon}{V}\right)^N}\right]. \quad (26)$$

This equation (26) is identical with the another work of Santos *et al* [22].

(vi) Again when $A = 0$ but $\alpha \neq 0$ & $n \neq 0$ the equation (24) gives

$$\left(\frac{\partial P}{\partial V}\right)_S = \frac{P}{3V(1+\alpha)} \left[\frac{3N\alpha \left(\frac{\epsilon}{V}\right)^N}{1 + \left(\frac{\epsilon}{V}\right)^N} - n\right]. \quad (27)$$

This is the case of Variable Generalised Chaplygin gas (VGCG) model [26].

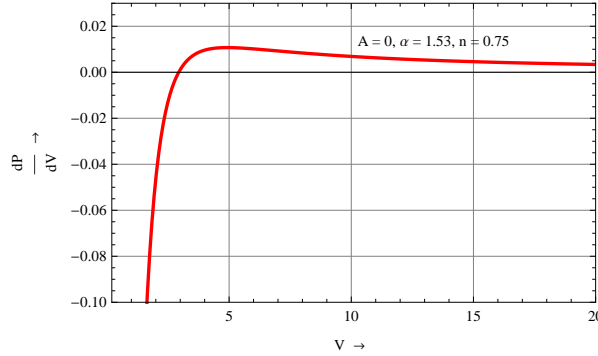


Figure 5: The variation of $(\frac{\partial P}{\partial V})_S$ and V is shown for $A = 0$, $\alpha = 1.53$, $n = 0.75$ and also $B_0 = 1$, $c = 1$.

It is clear from equation (27) that for $\alpha > 0$, n should be negative for $(\frac{\partial P}{\partial V})_S < 0$ throughout the evolution. It may not be out of place to call attention to an earlier work of Lu [26] where he studied the Union SNe Ia data and Sloan Digital Sky Survey (SDSS) baryon acoustic peak to constrain the Variable Generalised Chaplygin Gas (VGCG) model where he obtained the best fit values of $n = 0.75$ and $\alpha = 1.53$. Our analysis shows that Lu's conclusion is untenable if consideration of thermodynamical stability is taken into account. Similar conclusion may be drawn from the analysis of our fig-5. In this connection it may be pointed out that we have studied [27] for the case of $A = 0$ but $\alpha = 1$ & $n \neq 0$, where we have seen that $(\frac{\partial P}{\partial V})_S < 0$ for $n < 0$ throughout the evolution.

But our analysis is based on all these constants: A , B_0 , α and n . As it is not possible to constrain all the parameters simultaneously from equation (24), we have tried it step by step. Now we want to concentrate on the signature of n rather than the other parameters because we are dealing with VMCG model where n has a special role throughout the evolution, particularly at late time. Now $n = 0$ was discussed earlier by several authors [22]. It is seen from the equation (24) that for positive values of A , B_0 and α , it will be $(\frac{\partial P}{\partial V})_S < 0$ when $n < 0$ throughout the evolution. Fig-4 gives similar type of conclusion. Again it is seen from the fig-4 that for the positive value of n at small volume $(\frac{\partial P}{\partial V})_S < 0$ but as volume increases, i.e., at the late universe where the influence of n is significant, for $n > 0$, $(\frac{\partial P}{\partial V})_S > 0$; so the stability is questionable throughout the evolution for $n > 0$. Therefore, in the context of thermodynamical stability, we have to conclude that the value of n should be negative for the VMCG model.

Now we have to examine if $(\frac{\partial P}{\partial V})_T \leq 0$ as well. We will show in the next section that for $n < 0$ this condition may also be satisfied.

One should also verify the positivity of thermal capacity at constant volume c_v where $c_v = T(\frac{\partial S}{\partial T})_V = (\frac{\partial U}{\partial T})_V = V(\frac{\partial p}{\partial T})_V$. Now we determine the temperature T of the Variable modified Chaplygin gas as a function of its volume V and its entropy S . The temperature T of this fluid is determined from the relation $T = (\frac{\partial U}{\partial S})_V$. Using the above relation of the temperature and with the help of equation (6) we get the expression of T as

$$T = \frac{V^{1-N-\frac{n}{3(1+\alpha)}}}{1+\alpha} \left[\frac{B_0(1+\alpha)}{N} + V^{-N}c \right]^{-\frac{\alpha}{1+\alpha}} \left(\frac{\partial c}{\partial S} \right)_V \quad (28a)$$

$$= \frac{V^{1-N-\frac{n}{3}}}{1+\alpha} \rho^{-\alpha} \left(\frac{\partial c}{\partial S} \right)_V. \quad (28b)$$

If c is also assumed to be a universal constant, then $\frac{dc}{dS} = 0$ and the fluid, in such a condition, remains at zero temperature for any value of its volume and pressure. Therefore, to discuss extensively the thermodynamic stability of the variable modified Chaplygin gas whose temperature varies during its expansion, it is necessary to assume that the derivative of equation (28) is not zero implying $\left(\frac{\partial c}{\partial S}\right) \neq 0$. We have no *a priori* knowledge of the functional dependence of c . From physical considerations, however, we know that this function must be such as to give positive temperature and cooling along an adiabatic expansion and so we choose that $\left(\frac{\partial c}{\partial S}\right) > 0$.

Now from dimensional analysis, we observe from equation (6) that

$$[U] = \left\{ \frac{[c]}{[V]^{A(1+\alpha)}} \right\}^{\frac{1}{1+\alpha}}. \quad (29)$$

Since $[U] = [T][S]$, we can write

$$[c] = [T]^{1+\alpha} [S]^{1+\alpha} [V]^{A(1+\alpha)}. \quad (30)$$

It is difficult to get an analytic solution of c from equation (30), so as a trial case, we take an empirical expression of c and then to check if the resulting expressions satisfy standard relations of thermodynamics. But as c is a function of entropy only, the expression of c will be

$$c = (\tau v^A)^{1+\alpha} S^{1+\alpha}, \quad (31)$$

where τ and v are constants having the dimensions of time and volume respectively. Now

$$\frac{dc}{dS} = (1+\alpha) (\tau v^A)^{1+\alpha} S^\alpha. \quad (32)$$

Using equation (28) and (32), we get the expression of temperature

$$T = V^{1-N-\frac{n}{3(1+\alpha)}} \left[\frac{B_0(1+\alpha)}{N} + V^{-N}c \right]^{-\frac{\alpha}{1+\alpha}} (\tau v^A)^{1+\alpha} S^\alpha \quad (33a)$$

$$= V^{1-N-\frac{n}{3}} (\tau v^A)^{1+\alpha} \rho^{-\alpha} S^\alpha \quad (33b)$$

$$= \frac{\tau v^A}{V^A} \left\{ 1 - \frac{1}{1 + \left(\frac{\epsilon}{V}\right)^N} \right\}^{\frac{\alpha}{1+\alpha}}, \quad (33c)$$

and from equation (33a), the entropy is

$$S = \frac{\left[\frac{B_0(1+\alpha)}{N} V^N \right]^{\frac{1}{1+\alpha}} \left(\frac{T}{\tau^{1+\alpha}} \right)^{\frac{1}{\alpha}} \left(\frac{V}{v^{1+\alpha}} \right)^{\frac{A}{\alpha}}}{\left\{ 1 - \left(\frac{T}{\tau} \right)^{\frac{1+\alpha}{\alpha}} \left(\frac{V}{v} \right)^{\frac{A(1+\alpha)}{\alpha}} \right\}^{\frac{1}{1+\alpha}}}, \quad (34)$$

It follows from equation (34) that for positive and finite entropy one should have $0 < TV^A < \tau v^A$, but individually $0 < T < \tau$ and $v < V < \infty$, *i.e.*, τ represents the maximum temperature whereas v represents the minimum volume. It is shown from the equation (33c) that as volume of the VMCG increases, temperature decreases. It is also proved from the equation (33c) that when $T \rightarrow 0$, $V \rightarrow \infty$ and when $T \rightarrow \tau$, $V \rightarrow v$. Thus we can apparently avoid the initial singularity.

Evidently at $T = 0$, $S = 0$ which implies that the third law of thermodynamics is satisfied in this case.

Now using equation (34) we get the expression of thermal heat capacity as

$$c_V = T \left(\frac{\partial S}{\partial T} \right)_V = \frac{\left[\frac{B_0(1+\alpha)}{N} V^N \right]^{\frac{1}{1+\alpha}} \left(\frac{T}{\tau^{1+\alpha}} \right)^{\frac{1}{\alpha}} \left(\frac{V}{v^{1+\alpha}} \right)^{\frac{A}{\alpha}}}{\alpha \left\{ 1 - \left(\frac{T}{\tau} \right)^{\frac{1+\alpha}{\alpha}} \left(\frac{V}{v} \right)^{\frac{A(1+\alpha)}{\alpha}} \right\}^{\frac{2+\alpha}{1+\alpha}}}. \quad (35)$$

Since $0 < TV^A < \tau v^A$ and $\alpha > 0$, $c_V > 0$ is always satisfied irrespective of the value of n . This ensures the positivity of α . It is interesting to note that when the temperature goes to zero c_V goes to zero as expected from the third law of thermodynamics.

If we put $A = 0$ and $\alpha = 1$, *i.e.* Variable Chaplygin gas model, we get the identical expression of c_V of our previous work [27]. Again for $A = 0$ and $n = 0$, the equation (35) reduces to the work of Santos et al [21].

To end the section a final remark may be in order. While *positivity* of specific heat is strongly desirable *vis-a-vis* when dealing with special relativity, in a recent communication Luongo *et al* [30] argued that in a FRW type of model like the one we are discussing a negative specific heat at constant volume and a vanishingly small specific heat at constant pressure (c_P) are compatible with observational data. In fact they have derived the most general cosmological model which is agreeable with the $c_V < 0$ and $c_P \sim 0$ values obtained for the specific heats of the universe and showed, in addition, that it also overcomes the fine-tuning and the coincidence problems of the Λ CDM model.

(f) Thermal EoS:

Since $P = P(V, T)$, using (6), (31) and (34) we get the internal energy as a function of both V and T as follows:

$$U = V \left\{ \frac{\frac{B_0(1+\alpha)}{N} V^{-\frac{n}{3}}}{1 - \left(\frac{T}{\tau} \right)^{\frac{1+\alpha}{\alpha}} \left(\frac{V}{v} \right)^{\frac{A(1+\alpha)}{\alpha}}} \right\}^{\frac{1}{1+\alpha}}. \quad (36)$$

Now using (2), (3) and (36) the Pressure is

$$P = \rho \left[A - \frac{N}{1+\alpha} \left\{ 1 - \left(\frac{T}{\tau} \right)^{\frac{1+\alpha}{\alpha}} \left(\frac{V}{v} \right)^{\frac{A(1+\alpha)}{\alpha}} \right\} \right]. \quad (37)$$

We have seen from equation (37) that for $A = 0$ and $\alpha = 1$, the solution reduces to our previous work [27]. It is to be seen that for $A = 0$ and $n = 0$ the above solution goes to an earlier work of Santos et al [21].

Now the thermal EoS parameter is given by

$$\omega = A - \frac{N}{1+\alpha} \left\{ 1 - \left(\frac{T}{\tau} \right)^{\frac{1+\alpha}{\alpha}} \left(\frac{V}{v} \right)^{\frac{A(1+\alpha)}{\alpha}} \right\}. \quad (38)$$

Thus thermal EoS parameter is, in general, a function of both temperature and volume. If we consider early stage of the universe when at very high temperature and small volume, *i.e.*, at $T \rightarrow \tau$ and also $V \rightarrow v$, we get from equation (38) that $\omega \approx A$, *i.e.*, $P \approx A\rho$. This is same as equation (16).

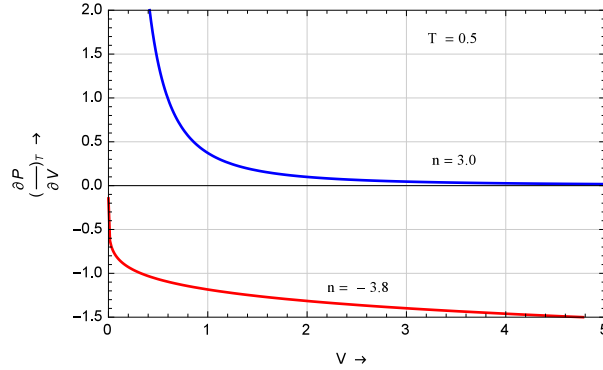


Figure 6: The variations of $(\frac{\partial P}{\partial V})_T$ vs V are shown. The graphs clearly show that $(\frac{\partial P}{\partial V})_T < 0$ throughout the evolution only for negative values of n . Here we have taken $A = 0.1$, $\alpha = 0.1$, $B_0 = 1$, $\tau = 1$ and $v = 1$.

Secondly we consider for large volume, *i.e.*, very low temperature, *i.e.*, $T \rightarrow 0$, we get from equation (38) that $\omega \approx -1 + \frac{n}{3(1+\alpha)}$, which is identical with equation (21) as is customary with the existing literature in this field [21,27]. Thus thermodynamical state represented by equations (16) and (17) are essentially same at both early and the late stage of the universe.

Now from equation (37) we have to examine if $(\frac{\partial P}{\partial V})_T \leq 0$, and a lengthy but straight forward calculation shows that only for negative value of n , this condition satisfies. It is very difficult to find out $(\frac{\partial P}{\partial V})_T$ from the equation (37) in a compact form and to get requisite inferences from it. So we have taken recourse to graphical method instead. From fig - 6, we clearly infer that $(\frac{\partial P}{\partial V})_T < 0$ for $n < 0$ throughout the evolution depending on the values of constants.

Now to discuss the thermodynamic stability of the VMCG one should have $(\frac{\partial P}{\partial V})_T < 0$ and $(\frac{\partial P}{\partial V})_S < 0$ *i.e.*, both isothermal and adiabatic situations need to be addressed. In the process we have found that $(\frac{\partial P}{\partial V})_S$ and $(\frac{\partial P}{\partial V})_T$ are negative for $n < 0$. Relevant to mention that in a previous work by Santos et al [22] it

was assumed $\left(\frac{\partial P}{\partial V}\right)_T = 0$ for simplicity, but in our case we have not made such a simplistic approach. In fact we have used a different type of approach to find out the expression of $c = c(S)$. As we have not made this assumption our analysis is more general in nature and our formulations would not reduce to there case when $n = 0$.

As pointed out in the previous section we here take up a standard relation of thermodynamics (e.g. the first internal energy equation) [31] as a trial case to see if the equation (31) is satisfied.

$$\left(\frac{\partial U}{\partial V}\right)_T = T \left(\frac{\partial P}{\partial T}\right)_V - P. \quad (39)$$

Using equations (36) and (37) we verify the relation (39) which proves the correctness of our approach.

(f) Pressure-Volume relation:

It is very difficult to arrive at a $(P \sim V)$ relation in a general way. So one has to take recourse to extremal conditions.

1. For small volume, $V \ll \epsilon$, which gives $P \sim A\rho$. In this case energy density and also pressure are very high. Using (33c) the temperature will be

$$T \approx \frac{\tau v^A}{V^A}. \quad (40)$$

At the early stage of the universe $V \rightarrow v$ (minimum volume) implies that $T \rightarrow \tau$ (maximum temperature) So the temperature is high enough at this stage. Using equations (33c) and (40) we get

$$\rho \approx S \frac{\tau v^A}{V^{1+A}}, \quad (41)$$

therefore,

$$UV^A = \rho V^{A+1} = S\tau v^A. \quad (42)$$

We know at this stage that $P \sim A\rho = A\frac{U}{V}$, *i.e.*, $PV \sim AU$, which gives using equation (42) $PV^{1+A} = S\tau v^A$. Since the entropy remains constant in an adiabatic process, this relation leads to $PV^{1+A} = \text{Constant}$. So it is observed that for small volume, *i.e.*, at high temperature the VMCG behaves as a fluid of $\gamma (= \frac{c_P}{c_V}) = 1 + A$. We can also rewrite the EoS as $P = (\gamma - 1)\rho$. Since early universe is radiation dominated *i.e.*, $A = \frac{1}{3}$, the value of $\gamma = 1 + A = \frac{4}{3}$, the pressure is related to the volume as $PV^{\frac{4}{3}} = \text{constant}$. Thus the VMCG behaves like a photon gas. The equation of adiabatic photon gas coincides with extreme relativistic electron gas.

2. For large volume, $V \gg \epsilon$,

Due to low density at this stage, entropy density is sufficiently small at the low temperature. We know from equation (17) that

$$P \approx \left\{ -1 + \frac{n}{3(1+\alpha)} \right\} \rho. \quad (43)$$

Again, from equation (9b), in this case we get

$$\rho \approx \left\{ \frac{B_0(1+\alpha)V^{-\frac{n}{3}}}{N} \right\}^{\frac{1}{1+\alpha}}. \quad (44)$$

Now using equations (43) and (44), we get

$$PV^{\frac{n}{3(1+\alpha)}} = \left\{ -1 + \frac{n}{3(1+\alpha)} \right\} \left\{ \frac{B_0(1+\alpha)}{N} \right\}^{\frac{1}{1+\alpha}}. \quad (45)$$

Equations (43), (44) and (45) can be obtained from the equations (38), (36) and (37) respectively at $T \rightarrow 0$. For an adiabatic system, the above equation looks like $PV^\gamma = k$, where k is a constant. Since we know that VMCG is thermodynamically stable for $n < 0$ which leads to $k < 0$. In this case, P is also negative. At the late stage of evolution, *i.e.*, at low temperature, the VMCG behaves like a fluid of $\gamma = \frac{n}{3(1+\alpha)}$. Interestingly, it is to be noted that the value of this γ depends on both n and α at this stage of evolution. It further follows from equation (33c) that $T \rightarrow 0$ for $\frac{\epsilon}{V} \ll 1$. Since any system near $T = 0$ is in states very close to its ground state, quantum mechanics is essential to the understanding of its properties. Indeed, the degree of randomness at these low temperatures is so small that discrete quantum effects can be observed on a macroscopic scale.

3. Discussion

Following the discovery of late acceleration of the universe there is a proliferation of varied dark energy models as its possible rescuers. While many of them significantly explain the observational findings coming out of different cosmic probes serious considerations have not been directed so far to the query whether the models are thermodynamically viable, for example if they obey the time honoured stability criteria. We have here considered a very general type of exotic fluid, termed ‘Variable Modified Chaplygin gas’ and studied its cosmological implications, mainly its thermodynamical stability. Regarding the cosmological dynamics we have come across two characteristic volumes of the fluid - V_c and V_f representing critical volume and flip volume respectively. The former refers to the case where pressure changes its sign while the latter gives the volume when the acceleration flip occurs. From physical consideration one should get $V_c < V_f$ - which also matches with our analysis. As discussed at the end of section 2(c), for flip to occur in FRW cosmology only a negative pressure is not the necessary and sufficient condition. The magnitude of pressure should be also less than $\frac{\rho}{3}$, which also follows from our analysis.

Although the exhaustive analysis of the latest cosmological observations provides a definite clue of the existence of dark energy in the universe but it is difficult to distinguish between the merits of various forms of dark energy at present. For the stability criteria we have followed the standard prescription : $(\frac{\partial P}{\partial V})_S < 0$, $(\frac{\partial P}{\partial V})_T < 0$ and $c_V > 0$. This, however, dictates that the new parameter, n introduced VMCG should be negative definite. This contrasts sharply with an earlier contention of Lu’s [26] where to explain the observational results they have to choose n as positive definite. This is pathological because it makes the system thermodynamically unstable.

Again this model shows that at early stage, the EoS becomes $P = A\rho$, where $0 \leq A \leq 1$. But at late stage, it reduces to equation (43). From the thermodynamical stability conditions, we find that $n < 0$, which favours a phantom like evolution and big rip is thus unavoidable. So far as phantom model is concerned, it is found to be compatible with SNe Ia observations and CMB anisotropy measurements [28]. The most important conclusion coming out of our analysis is that this model covers both big bang and big rip in the whole evolution process.

It is to be noted that at $T = 0$ the entropy of VMCG vanishes as in conformity with the third law of thermodynamics. We have studied both the thermal and the caloric EoS which shows that both $0 < T < \tau$ and $v < V < \infty$ where τ is a maximum temperature and v is a minimum volume attainable. Here τ and v are canonical in the sense that as $T \rightarrow \tau$, $V \rightarrow v$ at the early stage, *i.e.*, τ represents the maximum temperature that our VMCG model can sustain for the small finite volume, v .

We have also discussed ($P \sim V$) relation and at early stage ($A = \frac{1}{3}$) it is shown that for an isentropic system VMCG behaves like a photon gas, as it was at the time of radiation dominated era. It is also shown that for large volume, *i.e.*, at low temperature, entropy is sufficiently low which agrees well with the currently available low energy density of the universe.

Finally it may not be out of place to point out that as the field equations are very involved in nature we have to adopt an *ansatz* to determine the function of integration $c = c(S)$. However to justify it we have checked an important relation (39) (first internal energy equation) and have found it to be correct. So our *ansatz* is essentially viable. Moreover, this model is very general in the sense that many of earlier works in this field may be obtained as a special case.

To end a final remark may be in order. We have here concentrated on the cosmological and thermodynamical behaviour of the VMCG model mainly on a theoretical premise. However, as a future exercise one should try to constrain the value of the parameters associated with both thermal and caloric EoS in the light of observational values.

Acknowledgment : One of us(SC) acknowledges the financial support of UGC, New Delhi for a MRP award and acknowledges CERN for a short visit. DP acknowledges the financial support of UGC, ERO for a MRP (No- F-PSW- 165/13-14) and also acknowledges IRC,NBU for short visit. The authors wish to thank the anonymous referee for valuable comments and suggestions.

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